Illumination by Taylor Polynomials

Alan Horwitz

(Received 22 November 1999)

Abstract

Let $f(x)$ be a differentiable function on the real line R, and let P be a point not on the graph of $f(x)$. Define the illumination index of P to be the number of distinct tangents to the graph of f which pass thru P. We prove that if f'' is continuous and nonnegative on R, $f'' \geq m > 0$ outside a closed interval of R, and f'' has finitely many zeros on R, then any point P *below* the graph of f has illumination index 2. This result fails in general if f'' is not bounded away from 0 on R. Also, if f'' has finitely many zeros and f'' is not nonnnegative on R, then some point below the graph has illumination index not equal to 2. Finally, we generalize our results to illumination by odd order Taylor polynomials.

2000 Mathematics Subject Classification: 26A06

1 Introduction

The central problem in differential calculus is to find the tangent line to a given curve $y = f(x)$ at a given point $(c, f(c))$ *on the graph of* f. A somewhat more complicated problem is: Given a point $P = (s, t)$ *not on the graph of* f, find all values of c so that the tangent line to the graph of f at $(c, f(c))$ passes through P. If such a c exists, we say that the point $(c, f(c))$ *illuminates* P. A typical example is: find all tangents to $y = x^2$ which pass through the point $(2, 3)$. In this case each of the points $(1, 1)$ and $(3, 9)$ would illuminate P. Of course, it is certainly possible that *no* tangent line at all passes through the given point (s, t)–e.g. if $y = x^2$ and $P = (1, 3)$. A simple, but interesting exercise is: Let P be any point below the graph of $y = x^2$. Prove that there are exactly two tangents to the graph which pass through P. In considering

this type of problem, the following question naturally arises: given $f(x)$, for which points $P = (s, t)$ is there a tangent line to f which passes through P ? Also, how many tangents pass through P ? The questions above lead to some potentially interesting ideas for research. For example, suppose that f is convex on R, and let P be any point below the graph of $y = f(x)$. Are there always exactly two tangents to the graph which pass through P ? What if one assumes that $f''(x) > 0$ on R ? We give the answers in Section 1(see, in particular, Theorem 4).

In Section 2, we prove a converse result to Theorem 4. It is also natural to try to extend our results to illumination by *higher* order Taylor polynomials. In Section 3 we prove results similar to Theorem 4 for illumination by *odd* order Taylor polynomials. Most of the proofs extend verbatim, but some results from [2] are needed.

2 Illumination by Tangent Lines

Definition 1 *Let $f(x)$ be a differentiable function on the real line, and let P be any point not on the graph of f. We say that the illumination index of P is k if there are k distinct tangents to the graph of f which pass through P. We include the possibility that $k = \infty$.*

Remark 1 *We say that a tangent line T is multiple if T is tangent to the graph of f at more than one point. If only one tangent line T passes through P, but T is a multiple tangent, we still define the illumination index of P to be one. One could, of course, define an illumination index which takes into account the number of points of tangency of each tangent line.*

As noted earlier, any point below the graph of $y = x^2$ has illumination index 2. We now generalize this to convex C^2 functions in general, with the added condition that f'' is bounded below by a positive number outside some closed interval(see Theorem 4 below). First we prove a couple lemmas.

Lemma 2 *Let $f(x) \in C^2(R)$, and suppose that there exists $T > 0$ such that $f''(x) \geq m > 0$ on $|x| > T$. Let $T_c(x)$ be the tangent line to f at $(c, f(c))$. Then for any fixed s, $\lim\limits_{|c| \to \infty} T_c(s) = -\infty$.*

Proof. For fixed s, let $g(c) = T_c(s)$, which implies that $g'(c) = (s-c)f''(c)$. Let $U = \max(s,T)$, $u = \min(s,-T)$. It follows that $g'(c)$ is $\left\{\begin{matrix} \leq\ 0,\ c > U \\ \geq\ 0,\ c < u \end{matrix}\right\} \Rightarrow$ $g(c)$ is $\left\{\begin{matrix} \text{decreasing on } (U,\infty) \\ \text{increasing on } (-\infty,u) \end{matrix}\right\}$. Also, since $f''(x) \geq m > 0$ on $|x| > T$,

$$\left\{\begin{matrix} \lim\limits_{c\to\infty} g'(c) = -\infty \\ \lim\limits_{c\to-\infty} g'(c) = \infty \end{matrix}\right\} \tag{1}$$

Partition $[U,\infty)$ into infinitely many subintervals, $[c_{k-1}, c_k]$, of constant width $h > 0$. By (1), given $M > 0$, there exists $C > 0$ such that $g'(c) \leq -M$ for $c \geq C$. Now $g(c_k) = g(c_{k-1}) + \int\limits_{c_{k-1}}^{c_k} g'(t)dt \leq g(U) - Mh$ if $c_{k-1} \geq C$. Since this inequality holds for any $M > 0$, $g(c_k) \to -\infty$. Also, since the inequality holds for any increasing sequence $\{c_k\} \to \infty$, with $c_k - c_{k-1}$ constant, $\lim\limits_{c\to\infty} g(c) = -\infty$. A similar argument shows that $\lim\limits_{c\to-\infty} g(c) = -\infty$. ■

Remark 2 *Lemma 2 is a little easier to prove under the stronger assumption that $f''(x)$ is positive and bounded away from 0 on the real line. One can then just examine the error $E_c(x) = f(x) - T_c(x)$ and use Taylor's Remainder formula.*

Lemma 3 *Suppose that $f''(x)$ is continuous, nonnegative, and has finitely many zeros in R. Then at most two distinct tangent lines to f can pass through any given point P in the plane.*

Proof. Suppose that three distinct tangents, T_1, T_2, T_3 pass through P, and suppose that the T_i are tangent at $(x_i, f(x_i)), i = 1,2,3$. Assume, without loss of generality, that $x_1 < x_2 < x_3$. Since f is convex on any open interval, each pair of tangents has a unique point of intersection. Let I_1 equal the intersection point of T_1 and T_2,and let I_2 equal the intersection point of T_2 and T_3. Since all three tangents pass through P, $I_1 = I_2 = P$. If $I_1 = (s_1, t_1)$ and $I_2 = (s_2, t_2)$, then, again, since f is convex on any open interval, $x_1 < s_1 < x_2$ and $x_2 < s_2 < x_3$, which implies that $s_1 < s_2$,which contradicts the fact that $I_1 = I_2$.

Theorem 4 *Suppose that $f''(x)$ is continuous, nonnegative, and has finitely many zeros in R. Assume also that there exists $T > 0$ such that $f''(x) \geq m > 0$ on $|x| > T$. Let $P = (s,t)$ with $t < f(s)$. Then there are exactly two distinct tangent lines to the graph of f which pass through P.*

Proof. Since $t < f(s)$, for c sufficiently close to $s, T_c(s) = f(c)+f'(c)(s-c) > t$. By Lemma 2, $\lim\limits_{|c|\to\infty} T_c(s) = -\infty$. Hence, for $|c|$ sufficiently large, $T_c(s) < t$. By the Intermediate Value Theorem, $T_c(s) = t$ for at least two values of c. Note also that for a convex function, $c_1 \neq c_2 \Rightarrow T_{c_1} \neq T_{c_2}$. Hence the illumination index of P is at least two. By Lemma 3, the illumination index of P is at most two. That proves the theorem. ■

The following example shows that Theorem 4 does *not* hold in general for functions which only satisfy $f''(x) > 0$ on R.

Example 5 *Let* $f(x) = \int\limits_0^x \left(\int\limits_0^t e^{-u^2} du\right) dt = \frac{1}{2}\operatorname{erf}(x)\sqrt{\pi}x + \frac{1}{2}e^{-x^2} - \frac{1}{2}$, *where* $\operatorname{erf}(x) = \int\limits_0^x e^{-t^2} dt$. *Since* $f''(x) = e^{-x^2}$, $\lim\limits_{x\to\pm\infty} f''(x) = 0$. *We now show that no tangent to f passes through the point $(0,t)$ when $t < -\frac{1}{2} < f(0) = 0$. If the tangent line T_c to f passes through (s,t), then $f(c) + f'(c)(s-c) = t$. So consider the function* $h(x) = f(x) + f'(x)(s-x) - t = \frac{1}{2}e^{-x^2} - \frac{1}{2} + \frac{1}{2}\operatorname{erf}(x)\sqrt{\pi}s - t$. *If $s = 0$, then* $h(x) = \frac{1}{2}e^{-x^2} - \frac{1}{2} - t \Rightarrow h'(x) = -xe^{-x^2}$, *which implies that $h(x)$ is increasing for $x < 0$ and decreasing for $x > 0$. Since* $\lim\limits_{x\to\pm\infty}\left(\frac{1}{2}e^{-x^2} - \frac{1}{2} - t\right) = -\frac{1}{2} - t$, *if $t < -\frac{1}{2}$ then h is always positive and thus has no real zeroes.*

Our definition of the illumination index k includes the possibility that $k = \infty$. Of course, for polynomials the illumination index is always finite(indeed, it's bounded above by the degree of the polynomial). The following example shows that there are entire functions, however, where *almost every* point not on the graph has infinite illumination index.

Example 6 *Let $f(x) = \sin x$, and let $P = (s,t)$ be any point not on the graph of f, with $t \neq \pm 1$. The tangent line at $(c, f(c))$ passes through P if and only*

if $f(c) + f'(c)(s - c) = t$*, that is, when* $g(c) = \sin c + (s - c)\cos c - t = 0$*. For* n *sufficiently large and even,* $g(n\pi) = (-1)^n(s - n\pi) - t < 0$*, while for* n *sufficiently large and odd,* $g(n\pi) > 0$*. Hence* g *has infinitely many zeroes* $c_1, c_2, c_3, \ldots$ *Note that since* $t \neq \pm 1$*, none of the zeroes is an odd multiple of* $\frac{\pi}{2}$*, and hence none of the tangents at* $(c_j, \sin c_j)$ *is horizontal. Now each of these tangents passes through* P*, but they may not all be distinct. However, since a nonhorizontal line can only be tangent to* $y = \sin x$ *at finitely many points, it is clear that infintely many distinct tangents pass through* P*, and thus* P *has infinite illumination index.*

Remark 3 *Given* f*, one may define, for each nonnegative integer* k*, the set* D_k*, equal to the set of points in the plane with illumination index* k*. The* D_k *form a partition of* $R^2 - G$*, where* G *is the graph of* f*. For example, if* $f(x) = x^3$*, it is not hard to show that* $D_3 = \{(s,t) : s > 0, 0 < t < s^3\} \cup \{(s,t) : s < 0, s^3 < t < 0\}, D_2 = \{(s,t) : s \neq 0, t = 0\}, D_1 = G - (D_2 \cup D_3)$*, and* $D_k = \emptyset$ *for* $k = 0$ *or* $k > 3$*.*

3 A Converse Result

Suppose that $f(x)$ is *not* convex on R. Is it possible for every point below the graph of f to have illumination index 2 ? The answer is no, and thus we have the following partial converse of Theorem 4.

Theorem 7 *Let* $f \in C^3(R)$ *and suppose that* $f''(x)$ *has finitely many zeroes in* R*. If* $f''(x)$ *is not nonnegative on* R*, then there is a point* P *below the graph of* f *with illumination index not equal to* 2*.*

Proof. . If $f''(x) \leq 0$ on R, then clearly any point P below the graph of f has illumination index 0. Hence we may assume that there are real numbers s and u such that $f''(s) > 0, f''(u) = 0$, and $f''(x)$ changes sign at $x = u$, with $f'' \geq 0$ between s and u. We consider the case $s < u$, the other case being similar. Let $P = (s,t)$, with t to be chosen shortly. Now the tangent line at $(c, f(c))$ passes through P if and only if $f(c) - cf'(c) + f'(c)s = t$, which holds if and only if $h(c) = 0$, where

$$h(c) = f(c) - cf'(c) + f'(c)s$$

$h'(c) = (s-c)f''(c)$ and $h''(c) = (s-c)f'''(c) - f''(c)$. Note that $h(s) = f(s)$, $h'(s) = 0$, and $h''(s) = -f''(s) < 0$, so that $h(s)$ is a local maximum of $h(c)$. Since $h'(c) \le 0$ on (s,u) and $h'(c) \ge 0$ on $(u, u+\epsilon)$, $h(u)$ is a local minimum of $h(c)$. Note that $h(u) < h(s)$. Let T be the line $y = h(u)$, the tangent to h at $(u, h(u))$.

Case 1: T only intersects the graph of h at $(u, h(u))$.

Then let $t = h(u)$.

Case 2: T intersects the graph of h at some point $Q \ne (u, h(u))$. If a Q exists such that $h-T$ changes sign at Q, then $y = h(u)+\epsilon$ intersects the graph of h in at least three points for some $\epsilon > 0$. If no such Q exists, then h must have another local minimum at Q. Then $y = h(u) + \epsilon$ intersects the graph of h in at least four points for some $\epsilon > 0$. In either case, let $t = h(u) + \epsilon$, with ϵ chosen sufficiently small so that $h(u) + \epsilon < h(s)$. Since the zeroes of h correspond to values of c such that the tangent line at $(c, f(c))$ passes through P, for case two there are at least three such values of c. However, it is possible that some of the corresponding tangents could be multiple. It was shown in [3], however, that f can have only *finitely many* multiple tangent lines in any bounded interval. Also, since each tangent is tangent at only finitely many points, we can also choose ϵ sufficiently small so that none of the tangents corresponding to the zeroes of h is multiple. Thus at least three *distinct* tangents pass through P.

In each case covered, P lies below the graph of f since $h(s) = f(s)$. Hence the illumination index of P is either one or greater than or equal to three, and thus cannot equal two. ■

4 Illumination by Higher Order Taylor Polynomials

The results of the previous section can be extended to illumination by Taylor polynomials of order r, r odd. In certain ways, the odd order Taylor polynomials $P_c(x)$ behave like tangent lines. Suppose that $f \in C^{r+1}(-\infty, \infty)$, and let $P_c(x)$ denote the Taylor polynomial to f of order r at $x = c$. In [1] it was proved that if $f^{(r+1)}(x) \ne 0$ on $[a, b]$, then there is a unique u, $a < u < b$, such that $P_a(u) = P_b(u)$. This defines a mean $m(a, b) \equiv u$. We shall prove a slightly stronger version of this result. The method of proof is very similar to that used in [2], where further results and generalizations of the means

$m(a,b)$ were proved.

For the rest of this section we assume that r is an *odd* positive integer.

Let $E_c(x) = f(x) - P_c(x)$. By the integral form of the remainder, we have

$$E_c(x) = \frac{1}{r!} \int_c^x f^{(r+1)}(t)(x-t)^r dt \tag{2}$$

Lemma 8 *Suppose that $f^{(r+1)}(x)$ is continuous, nonnegative, and has finitely many zeros in $[a,b]$. Then $P_b - P_a$ has precisely one real zero c, $a < c < b$.*

Proof. By (2),

$$E_a(x) = \frac{1}{r!} \int_a^x f^{(r+1)}(t)(x-t)^r dt$$

$$E_b(x) = \frac{1}{r!} \int_x^b f^{(r+1)}(t)(t-x)^r dt$$

$E_a'(x) = \dfrac{1}{(r-1)!} \displaystyle\int_a^x f^{(r+1)}(t)(x-t)^{r-1} dt \Rightarrow E_a'(x) < 0$ for $x < a$ and $E_a'(x) > 0$ for $x > a$. Hence $E_a(x)$ is strictly increasing on (a,b). Similarly, $E_b(x)$ is strictly decreasing on (a,b). Since $E_a(a) = 0$ and $E_b(b) = 0$, there is a unique c, $a < c < b$, such that $E_b(c) - E_a(c) = 0$. This implies that $P_b(c) - P_a(c) = 0$. Now

$(E_b - E_a)(x) = -\displaystyle\int_a^b f^{(r+1)}(t)(x-t)^r dt \Rightarrow (E_b - E_a)'(x) = -r \displaystyle\int_a^b f^{(r+1)}(t)(x-t)^{r-1} dt \le 0$ for $x \in R$. Since $f^{(r+1)}$ has finitely many zeros, this implies that $E_b - E_a$ is strictly decreasing on R. Hence $E_b - E_a$ has precisely one real zero, which implies that $P_b - P_a$ has precisely one real zero c, $a < c < b$.

Lemma 9 *Suppose that $f^{(r+1)}(x)$ is continuous, nonnegative, and has finitely many zeros in $[a,b]$. Let P be any point in the xy plane. Then at most two distinct Taylor polynomials of order r at $x = c$, $a \le c \le b$, can pass through P.*

Proof. Suppose that three distinct Taylor polynomials of order r, P_{c_1}, P_{c_2}, and P_{c_3}, pass through $P = (s, t)$. Then $(P_{c_2} - P_{c_1})(s) = 0$ and $(P_{c_3} - P_{c_2})(s) = 0$. Without loss of generality, assume that $a \le c_1 < c_2 < c_3 \le b$. By Lemma 8, $c_1 < s < c_2$ and $c_2 < s < c_3$, which is a contradiction. Hence at most two distinct Taylor polynomials of order r can pass through P.

Lemma 10 *Let $f(x) \in C^{r+1}(R)$. Suppose that there exists $T > 0$ such that $f^{(r+1)}(x) \ge m > 0$ on $|x| > T$. Then for any fixed s, $\lim_{|c| \to \infty} P_c(s) = -\infty$.*

Proof. The proof is almost identical to that of Lemma 2, and we omit it. ■

Theorem 11 *Suppose that $f^{(r+1)}(x)$ is continuous, nonnegative, and has finitely many zeros in R. In addition, assume that there exists $T > 0$ such that $f^{(r+1)}(x) \ge m > 0$ on $|x| > T$. Let $P = (s, t)$ with $t < f(s)$. Then there are exactly two distinct Taylor polynomials of order r to the graph of f which pass through P.*

Proof. Since $t < f(s)$, for c sufficiently close to s,

$P_c(s) = f(s) + \sum_{k=1}^{r} \frac{f^{(k)}(c)}{k!}(s-c)^k > t$. By Lemma 10, $\lim_{|c| \to \infty} P_c(s) = -\infty$, and hence, for $|c|$ sufficiently large, $P_c(s) < t$. By the Intermediate Value Theorem, $P_c(s) = t$ for at least two values of c. Also, it is not hard to show that if $f^{(r+1)}(x) > 0$ on R, then $c_1 \ne c_2 \Rightarrow P_{c_1} \ne P_{c_2}$. Hence the illumination index of P is at least two. By Lemma 9, it is *at most* two. That proves the theorem. ■

Example 12 *Let $f(x) = e^x + x^4, P = (0,0), r = 3$. Then Theorem 11 applies, and the illumination index of P equals 2. We now verify this by estimating the actual values of c. Let $P_c(x)$ be the third order Taylor polynomial to f at $(c, f(c))$. Then $P_c(0) = e^c - e^c c + \frac{1}{2} e^c c^2 - \frac{1}{6} e^c c^3 - c^4 = 0$ has solutions $c_1 \approx -.9953$ and $c_2 \approx .9782$. Note that if $f(x) = e^x$ instead, then the illumination index of P equals 1. This does not contradict Theorem 11 since $f^{(iv)}(x) \to 0$ as $x \to -\infty$.*

References

[1] A. Horwitz, Means and Taylor polynomials, J. Math. Anal. Appl. 149(1990), 220-235.

[2] A. Horwitz, Means and averages of Taylor polynomials, J. Math. Anal. Appl. 176(1993), 404-412.

[3] A. Horwitz, Reconstructing a function from its set of tangent lines, American Mathematical Monthly 96(1989), 807-813.